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On the spectrum for the linear artificial compressible system

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1 Introduction

This article studies the incompressible Navier-Stokes system

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, && (1.1)\\ \partial_t \boldsymbol{v} - \nu\Delta \boldsymbol{v} + \boldsymbol{v}\cdot\nabla \boldsymbol{v} + \nabla p &= \boldsymbol{g}, && (1.2)\end{aligned}$$

and the artificial compressible system for (1.1)–(1.2):

$$\begin{aligned} \epsilon^2\partial_t p + \operatorname{div} \boldsymbol{v} &= 0, && (1.3)\\ \partial_t \boldsymbol{v} - \nu\Delta \boldsymbol{v} + \boldsymbol{v}\cdot\nabla \boldsymbol{v} + \nabla p &= \boldsymbol{g}. && (1.4)\end{aligned}$$

Here $\boldsymbol{v} = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$ and $p = p(x,t)$ denote the unknown velocity field and pressure, respectively, at time $t > 0$ and position $x \in \Omega$, where Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$; $\boldsymbol{g} = \boldsymbol{g}(x)$ is a given external force and $\epsilon > 0$ is a small parameter, called the artificial Mach number.

The system of equations (1.1)–(1.2) and (1.3)–(1.4) are considered under the boundary condition

$$\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_*. \qquad (1.5)$$

Here $\boldsymbol{v}_*$ is a given velocity field satisfying $\int_{\partial\Omega} \boldsymbol{v}_*\cdot\boldsymbol{n}\, dS = 0$, where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$.

It is easy to see that the set of stationary solutions of (1.1)–(1.2) is the same as that of (1.3)–(1.4). Since the incompressible system (1.1)–(1.2) is obtained from the artificial compressible one (1.3)–(1.4) as the limit $\epsilon \to 0$, one could expect that solutions of (1.1)–(1.2) would be approximated by solutions of (1.3)–(1.4) with $\epsilon \ll 1$. However, the limiting procedure is a singular limit, so it is not straightforward to conclude that stability properties of u_s as a solution of (1.1)–(1.2) are the same as those as a solution of (1.3)–(1.4) even if $0 < \epsilon \ll 1$. In [11, 12] it was discussed whether (1.3)–(1.4) gives a good approximation of (1.1)–(1.2), when $0 < \epsilon \ll 1$, from the view point of the stability of stationary solutions.

In this article we give a summary of the paper [12] on the relation of stability properties between stationary solutions of (1.1)–(1.2) and (1.3)–(1.4).

A. Chorin ([1, 2, 3]) proposed the artificial compressible system (1.3)–(1.4) to find numerically stationary solutions of the incompressible Navier-Stokes equation (1.1)–(1.2). As mentioned above, the set of stationary solutions of (1.1)–(1.2) is the same as that of (1.3)–(1.4). If solutions of the artificial compressible system (1.3)–(1.4) converge to a function $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ as $t \to \infty$, then the limit u_s is a stationary solution of (1.3)–(1.4), and thus, u_s is a stationary solution of (1.1)–(1.2). Chorin numerically obtained stationary cellular convection patterns of the Bénard convection problem described by the Oberbeck-Boussinesq equation

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, & (1.6)\\ \mathrm{Pr}^{-1}\left(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla \boldsymbol{v}\right) - \Delta \boldsymbol{v} + \nabla p - \sqrt{\mathrm{Ra}}\,\theta \boldsymbol{e}_3 &= \boldsymbol{0}, & (1.7)\\ \partial_t\theta + \boldsymbol{v}\cdot\nabla\theta - \Delta\theta - \sqrt{\mathrm{Ra}}\,\boldsymbol{v}\cdot \boldsymbol{e}_3 &= 0 & (1.8) \end{aligned}$$

in the infinite layer $\{x = (x', x_3); x' = (x_1, x_2) \in \mathbb{R}^2, 0 < x_3 < 1\}$ by using the corresponding artificial system

$$\begin{aligned} \epsilon^2\partial_t p + \operatorname{div} \boldsymbol{v} &= 0, & (1.9)\\ \mathrm{Pr}^{-1}\left(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla \boldsymbol{v}\right) - \Delta \boldsymbol{v} + \nabla p - \sqrt{\mathrm{Ra}}\,\theta \boldsymbol{e}_3 &= \boldsymbol{0}, & (1.10)\\ \partial_t\theta + \boldsymbol{v}\cdot\nabla\theta - \Delta\theta - \sqrt{\mathrm{Ra}}\,\boldsymbol{v}\cdot \boldsymbol{e}_3 &= 0. & (1.11) \end{aligned}$$

Here $\theta(x,t)$ is the temperature deviation from the heat conductive state; $\boldsymbol{e}_3 = {}^\top(0, 0, 1) \in \mathbb{R}^3$; $\mathrm{Pr} > 0$ and $\mathrm{Ra} > 0$ are non-dimensional parameters, called the Prandtl and Rayleigh numbers, respectively

In [11] the following questions were considered for (1.6)-(1.8) and (1.9)-(1.11):

(i) if u_s is stable as a solution of (1.9)-(1.11), then is u_s stable as a solution of (1.6)-(1.8) ? In other words, whether u_s represents an observable stationary flow in the real world ?

(ii) conversely, if u_s is stable as a solution of (1.6)-(1.8), then is u_s stable as a solution of (1.9)-(1.11) for $0 < \epsilon \ll 1$? In other words, what kind of stationary flows can be computed by Chorin's method ?

In [11], the above questions were considered for the Oberbeck-Boussinesq equation (1.6)–(1.8) in the infinite layer under the boundary condition $\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$ on $\{x_3 = 0, 1\}$ and a periodic boundary condition in $x' = (x_1, x_2)$. The results can be restated for the systems (1.1)–(1.2) and (1.3)–(1.4) in the following way.

We introduce the linearized operators around a stationary solution $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ associated with the systems (1.1)–(1.2) and (1.3)–(1.4) under (1.5). Here and in what follows ${}^\top\cdot$ stands for the transposition. Let $L : L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ be the operator defined by

$$L = -\nu \mathbb{P}\Delta + \mathbb{P}(\boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s))$$

with domain $D(L) = [H^2(\Omega) \cap H^1_0(\Omega)]^3 \cap L^2_\sigma(\Omega)$. Here $H^k(\Omega)$ denotes the k th order L^2-Sobolev space on Ω, $\mathbb{P}$ is the orthogonal projection, called the Helmholtz projection, from $L^2(\Omega)^3$ to $L^2_\sigma(\Omega)$, and $L^2_\sigma(\Omega)$ denotes the set of all L^2-vector fields $\boldsymbol{w}$ on Ω satisfying $\operatorname{div} \boldsymbol{w} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$, where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. We define the operator $L_\epsilon : H^1_*(\Omega) \times L^2(\Omega)^3 \to H^1_*(\Omega) \times L^2(\Omega)^3$, acting on $u = {}^\top(p, \boldsymbol{w})$, by

$$L_\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\operatorname{div} \\ \nabla & -\nu\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) \end{pmatrix}$$

with domain $D(L_\epsilon) = H^1_*(\Omega) \times [H^2(\Omega) \cap H^1_0(\Omega)]^3$. Here $H^1_*(\Omega)$ denotes the set of H^1 functions on Ω that have zero mean value over Ω.

Concerning the question (i), it was proved in [11] that if there exists a positive number b_0 such that $\rho(-L_{\epsilon_n}) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_0\}$ for some sequence $\epsilon_n \to 0$ as $n \to \infty$, then there exists a positive constant b_1 such that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_1\}$. Therefore, a stationary solution obtained by Chorin's method with $0 < \epsilon \ll 1$ is stable as a solution of the incompressible system (1.1)–(1.2). Furthermore, the instability result was proved: if $\sigma(-L) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$, then $\sigma(-L_\epsilon) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$ for $0 < \epsilon \ll 1$. This shows that unstable stationary solutions of (1.1)–(1.2) cannot be obtained by Chorin's method with $0 < \epsilon \ll 1$.

Concerning the question (ii), it was shown in [11] that if $\rho(-L) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_0\}$ for some positive constant b_0, then there exist positive constants δ_0 and b_1 such that $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -b_1\}$ for $0 < \epsilon \ll 1$, provided that

$$\inf_{\boldsymbol{w} \in H_0^1(\Omega)^3, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\mathrm{Re}\,(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w})_{L^2}}{\|\nabla \boldsymbol{w}\|_{L^2}^2} \geq -\delta_0. \tag{1.12}$$

This gives a sufficient condition for u_s to be computed by Chorin's method with $0 < \epsilon \ll 1$. The corresponding result for the Oberbeck-Boussinesq system (1.6)–(1.8) is stated exactly in the same form; and the result is applicable to stable bifurcating cellular convective patterns of the system (1.6)–(1.8), such as roll pattern, hexagonal pattern and etc., since they bifurcate from $\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$, and hence, the condition (1.12) is satisfied near the bifurcation point. However, the condition (1.12) seems to be somewhat stringent since most of its applications might be limited to stationary flows whose velocity fields are sufficiently small.

In [12] an improvement of the condition (1.12) was given. It was shown that the condition (1.12) can be replaced by

$$\inf_{\boldsymbol{w} \in H_0^1(\Omega)^3, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\mathrm{Re}\,((\mathbb{Q}\boldsymbol{w}) \cdot \nabla \boldsymbol{v}_s, \mathbb{Q}\boldsymbol{w})_{L^2}}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_{L^2}^2} \geq -\delta_0. \tag{1.13}$$

Here $\mathbb{Q} = I - \mathbb{P}$ is the orthogonal projection from $L^2(\Omega)^3$ to the space $G^2(\Omega) = \{\nabla p; p \in H_*^1(\Omega)\}$ which is the orthogonal complement of $L_\sigma^2(\Omega)$. The same result also holds for the case of the Oberbeck-Boussinesq system (1.6)–(1.8).

One can apply the condition (1.13) to the Taylor problem, namely, a flow between two concentric infinite cylinders, whose inner cylinder rotates with a uniform speed and outer one is at rest. It is well known that if the rotation speed is sufficiently small, then a laminar flow, called the Couette flow, is stable. When the rotation speed increases, beyond a certain value of the rotation speed, the Couette flow is getting unstable, and a vortex pattern is observed. The vortex pattern is periodic in the direction of the axis of the cylinders and it is called the Taylor vortex. The Taylor has been studied mathematically as a bifurcation problem for the incompressible system (1.1)–(1.2) (see [4, 9, 10, 13, 17]). The velocity field near the bifurcation point of the Taylor vortex is not necessarily small, and hence, it is unclear if the condition (1.12) can be applied to the Taylor vortex. However, it is not so difficult to show that the condition (1.13) is satisfied by the velocity field of the Taylor vortex under *axi-symmetric perturbations*. One can thus conclude that the Taylor vortex can be computed by Chorin's method. See [12, Secition 5] for the details.

We also note that the convergence of solutions as $\epsilon \to 0$ was discussed in [14, 15, 16] for the system (1.3)–(1.4) with the additional stabilizing nonlinear term $+\frac{1}{2}(\operatorname{div} \boldsymbol{v})\boldsymbol{v}$ on the left of (1.4). It was shown in [14, 15, 16] that there exists a weak solution ${}^\top(p_\epsilon, \boldsymbol{v}_\epsilon)$ for each $\epsilon > 0$ such that $\boldsymbol{v}_{\epsilon'} \to \boldsymbol{v}$ in $L^2(0,T; L^2(\Omega)^3)$ and $\nabla p_{\epsilon'} \to \nabla p$ weakly in $H^{-1}(\Omega \times (0,T))$ for all $T > 0$ along a sequence $\epsilon' \to 0$, where ${}^\top(p, \boldsymbol{v})$ is a weak solution of (1.1)–(1.2). We also mention the works by Donatelli [5, 6] and Donatelli and Marcati [7, 8] where similar convergence results were obtained in the case of unbounded domains by using the wave equation structure of the pressure and the dispersive estimates.

This article is organized as follows. In section 2 we state the result on the stability criterion obtained in [12]. In section 3 we give an outline of an proof of the result on the stability criterion, i.e., we outline that the condition (1.13) gives a stability criterion.

We close this section by introducing notation used in this article.

For $1 \le p \le \infty$ we denote by $L^p(D)$ the usual Lebesgue space over D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. The mth order L^2 Sobolev space over D is denoted by $H^m(D)$, and its norm is denoted by $\|\cdot\|_{H^m(D)}$. When $D = \Omega$, we simply denote these norms by $\|\cdot\|_p$, $\|\cdot\|_{H^m}$. The inner product of $L^2(D)$ is denoted by $(\cdot,\cdot)_{L^2(D)}$, i.e.,

$$(f,g)_{L^2(D)} = \int_D f(x)\overline{g(x)}dx.$$

Here $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. When $D = \Omega$ we simply denote $(\cdot,\cdot)_{L^2(D)}$ by $(\cdot,\cdot)$.

We set

$$\begin{aligned} H_0^1(D) &= \text{the } H^1(D)\text{-closure of } C_0^\infty(D), \\ H^{-1}(D) &= \text{the dual space of } H_0^1(D), \\ \dot{H}^1(D) &= \{f \in L^2_{loc}(D) : \|\nabla f\|_{L^2(D)} < \infty\}, \\ \dot{H}^{-1}(D) &= \text{the dual space of } \dot{H}^1(D). \end{aligned}$$

We define $L^2_*(\Omega)$ and $H^k_*(\Omega)$ by

$$L^2_*(\Omega) = \{f \in L^2(\Omega); \int_\Omega f(x)dx = 0\},$$

$$H_*^k(\Omega) = H^k(\Omega) \cap L_*^2(\Omega) \ \ (k \geq 1).$$

We set

$$L_\sigma^2(\Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}.$$

Here and in what follows, $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. It is known that $(L^2(\Omega))^3 = L_\sigma^2(\Omega) \oplus G^2(\Omega)$, where $G^2(\Omega) = \{\nabla p;\ p \in H_*^1(\Omega)\}$ is orthogonal complement of $L_\sigma^2(\Omega)$.

The orthogonal projection $\mathbb{P}$ from $L^2(\Omega)^3$ onto $L_\sigma^2(\Omega)$ is called the Helmholtz projection. We set $\mathbb{Q} = I - \mathbb{P}$.

We denote the resolvent set of an operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$.

2 Stability criterion

We state the stability criterion given in [12]. We introduce the linearized operators for the Navier-Stokes and the corresponding artificial compressible systems. Suppose that $u_s = {}^\top(p_s, \boldsymbol{v}_s)$ be a smooth stationary solution of (1.1)–(1.2), (1.5). Then, the perturbation equation takes the form

$$\begin{aligned} \mathrm{div}\,\boldsymbol{w} &= 0, & (2.1)\\ \partial_t\boldsymbol{w} - \nu\Delta\boldsymbol{w} + \boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s + \boldsymbol{w}\cdot\nabla\boldsymbol{w} + \nabla p &= \boldsymbol{0}. & (2.2)\end{aligned}$$

We consider (2.1)–(2.2) under the boundary condition

$$\boldsymbol{w}|_\Omega = \boldsymbol{0}. \tag{2.3}$$

Applying the Helmholtz projection $\mathbb{P}$ we have

$$\frac{d\boldsymbol{w}}{dt} + L\boldsymbol{w} + \mathbb{P}(\boldsymbol{w}\cdot\nabla\boldsymbol{w}) = \boldsymbol{0}, \tag{2.4}$$

where $L:\ L_\sigma^2(\Omega) \to L_\sigma^2(\Omega)$ denotes the linearized operator around $\boldsymbol{v}_s$ defined by

$$\begin{aligned} &D(L) = (H^2(\Omega) \cap H_0^1(\Omega))^3 \cap L_\sigma^2(\Omega),\\ &L\boldsymbol{w} = -\nu\mathbb{P}\Delta\boldsymbol{w} + \mathbb{P}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s) \ \ (\boldsymbol{w} \in D(L)).\end{aligned}$$

The corresponding artificial system takes the form

$$\frac{du}{dt} + L_\epsilon u + N(u, u) = 0. \tag{2.5}$$

Here $u = {}^{\top}(p, \boldsymbol{w})$; $L_\epsilon : H^1_*(\Omega) \times L^2(\Omega)^3 \to H^1_*(\Omega) \times L^2(\Omega)^3$ denotes the linearized operator around u_s defined by $H^1_*(\Omega) \times L^2(\Omega)^3$ defined by

$$D(L_\epsilon) = H^1_*(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))^3,$$
$$L_\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon^2}\mathrm{div} \\ \nabla & -\nu\Delta + \boldsymbol{v}_s \cdot \nabla + {}^{\top}(\nabla \boldsymbol{v}_s) \end{pmatrix};$$

and $N(u, u)$ is the nonlinear term given by

$$N(u, u) = {}^{\top}(0, \boldsymbol{w} \cdot \nabla \boldsymbol{w})$$

for $u = {}^{\top}(p, \boldsymbol{w})$.

The following result was obtained in [12].

Theorem 2.1. *([12]) Suppose that $\rho(-L) \supset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq -b_0\}$ for some positive constant b_0. Then there exist positive constants ϵ_0, δ_0 and b_1 such that if*

$$\inf_{\boldsymbol{w} \in H^1_0(\Omega)^3,\, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\mathrm{Re}\,((\mathbb{Q}\boldsymbol{w}) \cdot \nabla \boldsymbol{v}_s,\, \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_2^2} \geq -\delta_0, \tag{2.6}$$

then $\rho(-L_\epsilon) \supset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq -b_1\}$ for all $0 < \epsilon \leq \epsilon_0$.

Remark 2.2. *As an application of Theorem 2.1 (and [12, Rem. 2.2]), we mention the Taylor problem, a flow between concentric cylinders whose inner part rotates and the outer one is at rest. In fact, one can show that the bifurcating Taylor vortex is stable as a solution of the artificial compressible system for $0 < \epsilon \ll 1$ under axisymmetric perturbations. This implies that the Taylor vortex can be computed by Chorin's method since the Taylor vortex is axisymmetric. See [12, Section 5] for the details.*

Remark 2.3. *It is easily verified from the proofs of Theorem 2.1 and [11, Theorem 3.3] that the same result also holds for the case of the Oberbeck-Boussinesq system* (1.6)–(1.8).

3 Outline of proof of Theorem 2.1

Following [12] we give an outline of the proof of Theorem 2.1. We consider the resolvent problem for $-L_\epsilon$:

$$\lambda u + L_\epsilon u = F, \quad u = {}^{\top}(p, \boldsymbol{w}) \in D(L_\epsilon), \tag{3.1}$$

where $F = {}^\top(f, \boldsymbol{g}) \in H^1_*(\Omega) \times L^2(\Omega)^3$ is given. For simplicity we set $\nu = 1$. The problem (3.1) is rewritten as

$$\begin{aligned} \epsilon^2 \lambda p + \operatorname{div} \boldsymbol{w} &= \epsilon^2 f, && (3.2)\\ \lambda \boldsymbol{w} - \Delta \boldsymbol{w} + \boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s + \nabla p &= \boldsymbol{g}, && (3.3)\\ \boldsymbol{w}|_{\partial\Omega} &= \boldsymbol{0}. && (3.4) \end{aligned}$$

The assumption of Theorem 2.1 is that

$$\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0\} \tag{3.5}$$

for some positive constant b_0.

We see from the following two propositions that a part of the spectrum $\sigma(-L_\epsilon)$ near the imaginary axis may possibly lie only in a region $\operatorname{Im} \lambda = O(\epsilon^{-1})$ under the assumption (3.5).

Proposition 3.1. *There exist positive constants a and b such that $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -a\epsilon^2 |\operatorname{Im} \lambda|^2 + b\} \subset \rho(-L_\epsilon)$ for all $0 < \epsilon \leq 1$.*

One can prove Proposition 3.1 by the standard Matsumura-Nishida energy method as in the proof of [11, Proposition 6.1].

Proposition 3.2. *There exist positive numbers ϵ_1 and a_1 such that*

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -b_0, |\lambda| \leq a_1 \epsilon^{-1}\} \subset \rho(-L_\epsilon)$$

for all $0 < \epsilon \leq \epsilon_1$.

Proposition 3.2 can be proved by the same perturbation argument as in the proof of [11, Proposition 6.3].

One can see from Propositions 3.1 and 3.2 that Theorem 2.1 holds without the condition (2.6) if $\sqrt{b/a} < a_1$. In the case $\sqrt{b/a} \geq a_1$, for some range of λ near the imaginary axis with $\operatorname{Im} \lambda = O(\epsilon^{-1})$, we still need to consider if this range belongs to $\rho(-L_\epsilon)$ for $0 < \epsilon \ll 1$.

To this end, it suffices to deduce a priori estimate for solutions of (3.1) uniformly for $\lambda = \mu + i\frac{\eta}{\epsilon}$ with $-\mu_0 \leq \mu \leq \mu_1$ and $a_1/2 \leq |\eta| \leq 2\sqrt{b/a}$, where μ_0 and μ_1 are some positive constants. In fact, if we obtain such a uniform a priori estimate, then it follows that $\{\lambda = \mu + i\frac{\eta}{\epsilon}; -\mu_0 \leq \mu \leq \mu_1, a_1/2 \leq |\eta| \leq 2\sqrt{b/a}\} \subset \rho(-L_\epsilon)$ by a standard continuation argument since $\lambda = \pm i\frac{a_1}{\epsilon} \in \rho(-L_\epsilon)$ for $0 < \epsilon \leq \epsilon_1$ by Proposition 3.2. We will establish an appropriate a priori estimate under the condition (2.6).

It is easily seen that Theorem 2.1 follows from the following proposition.

Proposition 3.3. *Let* $\lambda = \mu + i\frac{\eta}{\epsilon}$ *with* $\mu, \eta \in \mathbb{R}$. *Suppose that* $u = {}^\top(p, \boldsymbol{w}) \in D(L_\epsilon)$ *is a solution of* (3.1). *For given positive numbers* μ_1 *and* η_* *there exist positive constants* δ_1 *and* $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \beta, \Omega)$ *such that if*

$$\inf\left\{\frac{\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi)}{\|\Delta\varphi\|_2^2}; \varphi \in H^2_*(\Omega),\ \varphi \neq 0,\ \frac{\partial\varphi}{\partial\boldsymbol{n}}|_{\partial\Omega} = 0\right\} \geq -\delta_1$$

and

$$-\frac{\beta^2}{128} \leq \mu \leq \mu_1,\ \eta_* \leq \eta \leq C'\epsilon^{-1},$$

then

$$(\eta^3 + \beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 \leq C\left\{(\eta + \frac{\epsilon^2}{\eta})\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^2}{\eta}\|f\|_{H^1}^2\right\}$$

for all $0 < \epsilon \leq C'\min\{1, \eta_*, \sqrt{\frac{\eta_*}{\mu_*}}, \frac{\eta_*}{\mu_*}, \eta_*\mu_*^{-\frac{2}{3}}, \sqrt{\frac{1}{\mu_*}}\}$ *with* $\mu_* = \max\{\frac{\beta^2}{128}, \mu_1\}$.

Idea of proof of Proposition 3.3. To illustrate the idea of the proof of Proposition 3.3, we consider the case $\mu = 0$, i.e.,

$$\lambda = i\frac{\eta}{\epsilon}.$$

The following estimate can be proved in a similar manner to the proof of [11, Prop. 6.5]. See also [12, Prop. 3.5].

Proposition 3.4. *Let* η_* *be a given positive number. Let* $u = {}^\top(p, \boldsymbol{w}) \in D(L_\epsilon)$ *be a solution of* (3.1) *with* $\lambda = i\frac{\eta}{\epsilon}$, $\eta \in \mathbb{R}$. *There exists a positive constant* $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \beta, \Omega)$ *such that if*

$$\epsilon \leq C'\min\{1, \eta_*\},\ \eta_* \leq \eta \leq \frac{1}{4\epsilon}$$

then

$$(\eta^3 + 2\beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 \leq -64\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) + C(\epsilon^2\eta^2 + \epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.$$

Here $\boldsymbol{G}_\lambda = \lambda\boldsymbol{g} - \nabla f$; *and* C *is a positive constant depending only on* $\|\boldsymbol{v}_s\|_{C^1}$ *and* Ω.

Proof. Let $u = {}^\top(p, \boldsymbol{v}) \in D(L_\epsilon)$ be a solution of (3.1). Then, by (3.2), we have

$$p = -\frac{1}{\epsilon^2\lambda}\mathrm{div}\,\boldsymbol{w} + \frac{1}{\lambda}f.$$

Substituting this into (3.3), we obtain

$$\epsilon^2\lambda^2 \boldsymbol{w} - \epsilon^2\lambda\Delta \boldsymbol{w} - \nabla \mathrm{div}\, \boldsymbol{w} + \epsilon^2\lambda(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) = \epsilon^2 \boldsymbol{G}_\lambda. \tag{3.6}$$

We take the inner product of (3.6) with $\boldsymbol{w}$. It follows that

$$\epsilon^2\lambda^2\|\boldsymbol{w}\|_2^2 + \epsilon^2\lambda\|\nabla \boldsymbol{w}\|_2^2 + \|\mathrm{div}\, \boldsymbol{w}\|_2^2 = -\epsilon^2\lambda(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w}) + \epsilon^2(\boldsymbol{G}_\lambda, \boldsymbol{w}). \tag{3.7}$$

Since $\lambda^2 = -\frac{\eta^2}{\epsilon^2}$, the real part of (3.7) yields

$$\begin{aligned} -\eta^2\|\boldsymbol{w}\|_2^2 + \|\mathrm{div}\, \boldsymbol{w}\|_2^2 = {} & \epsilon\eta \mathrm{Im}\,(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w}) \\ & + \epsilon^2 \mathrm{Re}\,(\boldsymbol{G}_\lambda, \boldsymbol{w}). \end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned} \eta^2\|\boldsymbol{w}\|_2^2 \le {} & (3 + \epsilon\eta)\|\nabla \boldsymbol{w}\|_2^2 + \epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla \boldsymbol{v}_s\|_\infty)\|\boldsymbol{w}\|_2^2 \\ & + \epsilon^2\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2. \end{aligned} \tag{3.9}$$

The imaginary part of (3.7) yields

$$\begin{aligned} \eta\|\nabla \boldsymbol{w}\|_2^2 &= -\eta \mathrm{Re}\,(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w}) + \epsilon \mathrm{Im}\,(\boldsymbol{G}_\lambda, \boldsymbol{w}) \\ &\le -\eta \mathrm{Re}\,(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w}) + \frac{1}{2}\eta\|\nabla \boldsymbol{w}\|_2^2 + \epsilon\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2, \end{aligned}$$

and hence,

$$\begin{aligned} & \frac{1}{2}\eta\|\nabla \boldsymbol{w}\|_2^2 \\ \le & \quad -\eta \mathrm{Re}\,(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w}) + \epsilon\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2. \end{aligned} \tag{3.10}$$

By (3.9) and (3.10), we have

$$\begin{aligned} & \frac{\eta^3}{12}\|\boldsymbol{w}\|_2^2 + \frac{\eta}{4}(1 - \epsilon\eta)\|\nabla \boldsymbol{w}\|_2^2 \\ \le & \quad -\eta \mathrm{Re}\,(\boldsymbol{w} \cdot \boldsymbol{v}_s, \boldsymbol{w}) \\ & \quad +C\left\{\epsilon\eta^2\|\nabla \boldsymbol{v}_s\|_\infty + \epsilon\eta^2\|\boldsymbol{v}_s\|_\infty^2\right\}\|\boldsymbol{w}\|_2^2 + C(\epsilon^2\eta + \epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2, \end{aligned}$$

and consequently, if $\eta \le \frac{1}{4\epsilon}$, then

$$\begin{aligned} & \left(\frac{\eta^3}{12} + \frac{1}{16}\beta^2\eta\right)\|\boldsymbol{w}\|_2^2 + \frac{1}{16}\eta\|\nabla \boldsymbol{w}\|_2^2 \\ \le & \quad -\eta \mathrm{Re}\,(\boldsymbol{w} \cdot \boldsymbol{v}_s, \boldsymbol{w}) \\ & \quad +C\left\{\epsilon\eta^2\|\nabla \boldsymbol{v}_s\|_\infty + \epsilon\eta^2\|\boldsymbol{v}_s\|_\infty^2\right\}\|\boldsymbol{w}\|_2^2 + C(\epsilon^2\eta + \epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2. \end{aligned}$$

Therefore, there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1})$ such that if

$$\epsilon \leq C' \min\{1, \eta_*\}, \ \eta_* \leq \eta \leq \frac{1}{4\epsilon},$$

then

$$\frac{1}{32}(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2+\frac{1}{16}\eta\|\nabla\boldsymbol{w}\|_2^2 \leq -\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{w}_s, \boldsymbol{w})+C(\epsilon^2\eta^2+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.$$

This completes the proof. □

We next estimate solenoidal part of $\boldsymbol{w}$.

Proposition 3.5. *Let η_* be given a positive number. Let $u = {}^\top(p, \boldsymbol{w})$ be a solution of* (3.1) *with $\lambda = i\frac{\eta}{\epsilon}$, $\eta \geq \eta_*$. If $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ is the Helmholtz decomposition of $\boldsymbol{w}$, then*

$$\begin{aligned}\|\boldsymbol{v}\|_2^2 \ \leq \ C\Bigg\{&\frac{\epsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\|\nabla\varphi\|_{H^1}^2 + \frac{\epsilon^2}{\eta^2}\|\nabla\varphi\|_{H^2}^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2 \\ &+\frac{\epsilon^2}{\eta^2}\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\Bigg\},\end{aligned}$$

$$\|\boldsymbol{v}\|_{H^2}^2 \leq C\left\{\frac{\eta^{\frac{3}{2}}}{\epsilon^{\frac{3}{2}}}\|\nabla\varphi\|_{H^1}^2 + \|\nabla\varphi\|_{H^2}^2 + \|\boldsymbol{g}\|_2^2 + \|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\right\}.$$

To prove Proposition 3.5 we apply the following estimate for the Stokes system with nonhomogeneous boundary data.

Lemma 3.6. *Suppose that ${}^\top(p, \boldsymbol{v}) \in H^1_*(\Omega) \times H^2(\Omega)$ is a solution of*

$$\left\{\begin{aligned}\mathrm{div}\,\boldsymbol{v} &= 0, \\ \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla p &= \boldsymbol{g}, \\ \boldsymbol{v}|_{\partial\Omega} &= \boldsymbol{\psi},\end{aligned}\right. \tag{3.11}$$

with $\lambda \in \{\lambda \in \mathbb{C};\, |\mathrm{arg}\lambda| \leq \pi - \omega\}$ for some $0 < \omega < \frac{\pi}{2}$, $\boldsymbol{g} \in L^2(\Omega)$ and $\boldsymbol{\psi} \in H^{\frac{3}{2}}(\partial\Omega)$ satisfying $\boldsymbol{\psi}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$. Then there exists a positive constant $C = C(\omega, \Omega)$ such that

$$|\lambda|\|\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{H^2} + \|p\|_{H^1} \leq C\{\|\boldsymbol{g}\|_2 + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)}\}.$$

A proof of Lemma can be found in [12, Section 4].

Proof of Proposition 3.5. Let $u = {}^\top(p, \boldsymbol{w}) \in D(L_\epsilon)$ be a solution of (3.2)–(3.4) and let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{w}$. Then, $\operatorname{div}\boldsymbol{v} = 0$, $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, $\frac{\partial\varphi}{\partial\boldsymbol{n}}\big|_{\partial\Omega} = \boldsymbol{w}\cdot\boldsymbol{n}|_{\partial\Omega}$ and $\int_\Omega \varphi dx = 0$. Since $\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{0}$, we see that

$$\frac{\partial\varphi}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = 0$$

and

$$\left\{\begin{array}{rcl} \operatorname{div}\boldsymbol{v} &=& 0, \\ \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla q &=& \boldsymbol{g} - (\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s), \\ \boldsymbol{v}|_{\partial\Omega} &=& -\nabla\varphi|_{\partial\Omega}. \end{array}\right.$$

Here

$$q = \lambda\varphi - \Delta\varphi + p.$$

Note that

$$\int_\Omega q\,dx = \int_\Omega (\lambda\varphi - \Delta\varphi + p)\,dx = -\int_{\partial\Omega}\frac{\partial\varphi}{\partial\boldsymbol{n}}\,d\sigma = 0.$$

Applying Lemma 3.6, one can obtain the desired estimates. □

The potential flow part $\nabla\varphi$ satisfies the following estimates.

Proposition 3.7. *Let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be as in Proposition 3.5. Then there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1})$ such that if $0 < \epsilon \le C'\min\{1, \eta_*\}$, the following estimates*

$$\|\Delta\varphi\|_2^2 \le C_1\left\{\eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\right\}, \tag{3.12}$$

$$\frac{1}{\eta^2}\|\nabla\Delta\varphi\|_2^2 \le C_1\left\{\eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^2\|\Delta\boldsymbol{v}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\right\}, \tag{3.13}$$

hold with $C_1 > 0$ independent of η_*, ϵ, and Ω.

Outline of proof of Proposition 3.7. Let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{w}$. Since $\operatorname{div}\boldsymbol{w} = \Delta\varphi$, we see from (3.8) that

$$\begin{aligned} \|\Delta\varphi\|_2^2 &= \eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\operatorname{Im}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) + \epsilon^2\operatorname{Re}(\boldsymbol{G}_\lambda, \boldsymbol{w}) \\ &\le \eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla\boldsymbol{v}_s\|_\infty)\|\boldsymbol{w}\|_2^2 \\ &\quad + \epsilon^2\operatorname{Re}(\boldsymbol{G}_\lambda, \boldsymbol{w}), \end{aligned}$$

and hence,

$$\begin{aligned}\|\Delta\varphi\|_2^2 &\leq \eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \epsilon\eta(\|\boldsymbol{v}_s\|_\infty^2 + \|\nabla\boldsymbol{v}_s\|_\infty)\|\boldsymbol{w}\|_2^2 \\ &\quad + \epsilon^2 \mathrm{Re}\,(\boldsymbol{G}_\lambda, \boldsymbol{w}).\end{aligned}$$

By using the Hölder and Poincaré inequalities one can obtain the desired estimate for $\|\Delta\phi\|_2^2$.

We next establish the estimate(3.13). We take the inner product of (3.6) with $-\nabla\Delta\varphi$ to obtain

$$\begin{aligned}&-\epsilon^2\lambda^2(\boldsymbol{w}, \nabla\Delta\varphi) + \epsilon^2\lambda(\Delta\boldsymbol{w}, \nabla\Delta\varphi) + \|\nabla\Delta\varphi\|_2^2 \\ &= \epsilon^2\lambda(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \nabla\Delta\varphi) - \epsilon^2(\boldsymbol{G}_\lambda, \nabla\Delta\varphi).\end{aligned} \tag{3.14}$$

Since $\boldsymbol{w}|_{\partial\Omega} = 0$ and $\mathrm{div}\,\boldsymbol{w} = \Delta\varphi$, we have

$$-\epsilon^2\lambda^2(\boldsymbol{w}, \nabla\Delta\varphi) = \epsilon^2\lambda^2(\mathrm{div}\,\boldsymbol{w}, \Delta\varphi) = \epsilon^2\lambda^2\|\Delta\varphi\|_2^2,$$

$$\epsilon^2\lambda(\Delta\boldsymbol{w}, \nabla\Delta\varphi) = \epsilon^2\lambda(\Delta\boldsymbol{v}, \nabla\Delta\varphi) + \epsilon^2\lambda\|\nabla\Delta\varphi\|_2^2.$$

Taking the real part of (3.14), we thus have

$$\begin{aligned}&-\eta^2\|\Delta\varphi\|_2^2 + \|\nabla\Delta\varphi\|_2^2 \\ &\leq \frac{1}{2}\|\nabla\Delta\varphi\|_2^2 + 3\epsilon^4|\lambda|^2\{\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\} \\ &\quad + \frac{3}{2}\epsilon^4\|\boldsymbol{G}_\lambda\|_2^2 + \frac{3}{2}\epsilon^4|\lambda|^2\|\Delta\boldsymbol{v}\|_2^2.\end{aligned}$$

This implies that, if $\lambda = i\frac{\eta}{\epsilon}$ with $\eta \geq \eta_*$, then

$$\begin{aligned}\frac{1}{2}\|\nabla\Delta\varphi\|_2^2 \leq \eta^2\|\Delta\varphi\|_2^2 &+ \frac{3}{2}\epsilon^2\eta^2\|\Delta\boldsymbol{v}\|_2^2 \\ &+ 3\epsilon^2\eta^2\{\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\} \\ &+ \frac{3}{2}\epsilon^4\|\boldsymbol{G}_\lambda\|_2^2.\end{aligned} \tag{3.15}$$

By (3.12) and (3.15), one can obtain the desired estimate (3.13). See [12] for the details. $\square$

We are now in a position to prove Proposition 3.3 for the case $\lambda = i\frac{\eta}{\epsilon}$.

Proof of Proposition 3.3. Let $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{w}$. Then

$$\begin{aligned}
-\eta \mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s, \boldsymbol{w}) &\le -\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s,\nabla\varphi) + \eta\{|\mathrm{Re}\,(\boldsymbol{v}\cdot\nabla\boldsymbol{v}_s,\nabla\varphi)| \\
&\quad +|\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s,\boldsymbol{v})| + |\mathrm{Re}\,(\boldsymbol{v}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{v})|\} \\
&\le -\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s,\nabla\varphi) \\
&\quad +\kappa\eta\|\nabla\boldsymbol{v}_s\|_\infty\|\nabla\varphi\|_2^2 + (1+\frac{1}{\kappa})\eta\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{v}\|_2^2
\end{aligned}$$

for any $\kappa > 0$. Choose $\kappa = \frac{\beta^2}{64\|\nabla\boldsymbol{v}_s\|_\infty}$. Then, since $\|\boldsymbol{w}\|_2^2 = \|\boldsymbol{v}\|_2^2 + \|\nabla\varphi\|_2^2$, we see from Proposition 3.4 that

$$\begin{aligned}
&(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 \\
&\quad \le -c_0\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s,\nabla\varphi) + C\eta\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{v}\|_2^2 + C(\epsilon^2\eta^2+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2,
\end{aligned} \tag{3.16}$$

where $c_0 = 64$.

To compute the proof, we need to estimate the second term on the right-hand side of (3.16). Applying Propositions 3.5 and 3.7, we see that there exists a positive constant $C = C(\Omega)$ such that

$$\begin{aligned}
\frac{1}{\eta^2}\|\nabla\Delta\varphi\|_2^2 &\le C\Big\{\eta^2\|\boldsymbol{w}\|_2^2 + \epsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^{\frac12}\eta^{\frac32}\|\nabla\varphi\|_{H^1}^2 + \epsilon^2\|\nabla\Delta\varphi\|_2^2 \\
&\quad +\epsilon^2\|\boldsymbol{g}\|_2^2 + \epsilon^2\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2 \\
&\quad +\frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\Big\}.
\end{aligned}$$

It follows that if $\eta^2 \le \frac{1}{2C\epsilon^2}$, then

$$\begin{aligned}
\frac{1}{\eta^2}\|\nabla\Delta\varphi\|_2^2 &\le C\{(\eta^2+\epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2)\|\boldsymbol{w}\|_2^2 + (\epsilon\eta+\epsilon^2\|\boldsymbol{v}_s\|_\infty^2)\|\nabla\boldsymbol{w}\|_2^2 \\
&\quad +\epsilon^{\frac12}\eta^{\frac32}\|\nabla\varphi\|_{H^1}^2 + \epsilon^2\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^4}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\Big\}.
\end{aligned} \tag{3.17}$$

Using (3.17), Proposition 3.5 and the elliptic estimates: $\|\nabla\varphi\|_{H^k} \le C\|\nabla^{k-1}\Delta\varphi\|_2$ $(k=1,2)$, we obtain

$$\begin{aligned}
\|\boldsymbol{v}\|_2^2 &\le C\Big\{\Big(\frac{\epsilon^{\frac12}}{\eta^{\frac12}} + \epsilon^{\frac52}\eta^{\frac32}\Big)\|\Delta\varphi\|_2^2 + \epsilon^2(\eta^2+\epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2)\|\boldsymbol{w}\|_2^2 \\
&\quad +\epsilon^2(\epsilon\eta+\epsilon^2\|\boldsymbol{v}_s\|_\infty^2)\|\nabla\boldsymbol{w}\|_2^2 + \epsilon^4\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^6}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2 \\
&\quad +\frac{\epsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{v}_s\|_\infty^2\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\nabla\boldsymbol{v}_s\|_\infty^2\|\boldsymbol{w}\|_2^2\Big\}.
\end{aligned} \tag{3.18}$$

Furthermore, we see from Proposition 3.7 that

$$\eta\|\Delta\varphi\|_2^2 \le C_1\left\{\eta^3\|\boldsymbol{w}\|_2^2 + \epsilon\eta^2\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{\eta}\|\boldsymbol{G}_\lambda\|_2^2\right\}. \tag{3.19}$$

Combining (3.16), (3.18) and (3.19), we have

$$\begin{aligned}
&(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla\boldsymbol{w}\|_2^2 + \frac{\eta}{2C_1}\|\Delta\varphi\|_2^2\\
\le\ & -c_0\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi)\\
& +\frac{1}{2}\eta^3\|\boldsymbol{w}\|_2^2 + \frac{\epsilon}{2}\eta^2\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^4}{2\eta}\|\boldsymbol{G}_\lambda\|_2^2\\
& +C\eta\|\nabla\boldsymbol{v}_s\|_\infty\Big[\Big(\epsilon^{\frac{1}{2}}\eta^{-\frac{1}{2}} + \epsilon^{\frac{5}{2}}\eta^{\frac{3}{2}}\Big)\|\Delta\varphi\|_2^2\\
& +\epsilon^2(\eta^2+\epsilon^2\|\nabla\boldsymbol{v}_s\|_\infty^2)\|\boldsymbol{w}\|_2^2\\
& +\epsilon^2(\epsilon\eta+\epsilon^2\|\boldsymbol{v}_s\|_\infty^2)\|\nabla\boldsymbol{w}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{v}_s\|_\infty\|\nabla\boldsymbol{w}\|_2^2\\
& +\frac{\epsilon^2}{\eta^2}\|\nabla\boldsymbol{v}_s\|_\infty\|\boldsymbol{w}\|_2^2 + \epsilon^4\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^6}{\eta^2}\|\boldsymbol{G}_\lambda\|_2^2\Big]\\
& +C(\epsilon^2\eta^2+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2.
\end{aligned}$$

It then follows that there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \beta, \Omega)$ such that if $\epsilon \le C'\min\{1,\, \eta_*\}$, $\eta \le \frac{C}{\epsilon}$, then

$$\begin{aligned}
&\frac{1}{4}(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2 + \frac{1}{2}\eta\|\nabla\boldsymbol{w}\|_2^2 + \frac{1}{4C_1}\eta\|\Delta\varphi\|_2^2\\
&\le -c_0\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi) + C\epsilon^4\eta\|\boldsymbol{g}\|_2^2 + C(\epsilon^2\eta^2+\epsilon)\|\boldsymbol{G}_\lambda\|_2\|\boldsymbol{w}\|_2\\
&\quad +C\frac{\epsilon^4}{\eta}\|\boldsymbol{G}_\lambda\|_2^2 + C\frac{\epsilon^2}{\eta}\|\boldsymbol{g}\|_2^2.\\
&\le \frac{1}{8}(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2 - c_0\eta\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi)\\
&\quad +C\Big(\frac{\epsilon^4}{\eta} + \epsilon^4\eta + \frac{\epsilon^2}{\beta^2\eta}\Big)\|\boldsymbol{G}_\lambda\|_2^2 + C\Big(\epsilon^4\eta + \frac{\epsilon^2}{\eta}\Big)\|\boldsymbol{g}\|_2^2.
\end{aligned}$$

This implies that if

$$\inf\left\{\frac{\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s, \nabla\varphi)}{\|\Delta\varphi\|_2^2}; \varphi\in H_*^2(\Omega),\ \varphi\ne 0,\ \frac{\partial\varphi}{\partial\boldsymbol{n}}\Big|_{\partial\Omega} = 0\right\} \ge -\frac{1}{8c_0C_1},$$

then

$$(\eta^3 + \beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla \boldsymbol{w}\|_2^2 + \eta\|\Delta\varphi\|_2^2 \leq C\Big\{\Big(\eta + \frac{\epsilon^2}{\eta}\Big)\|\boldsymbol{g}\|_2^2 + \frac{\epsilon^2}{\eta}\|\nabla f\|_2^2\Big\}.$$

This completes the proof. □

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